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The Multitype Branching Diffusion

B. GAIL IVANOFF*

University of Ottawa, Ottawa, Ontario, Canada

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The multitype branching diffusion (MBD) is considered. A review of the general theory of multitype point processes is given in Section 2, and spatial central limit theorems for homogeneous infinitely divisible processes are proven in Section 3. In Section 4, the MBD is defined, and equations for its first four factorial moment density functions are found. The behaviour of the mean and covariance functionals as time approaches infinity is studied. The MBD with immigration (MBDI) is introduced in Section 5. The existence of a steady state is proven, and spatial central limit theorems are developed for the MBDI.

1. INTRODUCTION

The single-type branching diffusion process and variants of it have been studied extensively by Sawyer [11], Dawson and Ivanoff [4], Fleischman [5], and Ivanoff [6]. The purpose of this paper is to study the properties of the multitype branching diffusion process; this is a process in which there is more than one type of particle, each of which branches and diffuses independently in a domain D. Mutations from one type to another will be allowed. Sawyer [11] developed formulas for the first and second factorial moment densities for one type of particle in the case of a two-particle system. The multivariate probability-generating functional (PGF) will be used here to provide expressions for higher-order moments. The process with immigration will be studied in a manner analogous to Ivanoff [6].

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2. Results for Multitype Point Processes

Some of the properties of multitype point processes (m.p.p.) will be studied. There are two approaches to the theory of multitype point processes: the first approach generalizes the theory of single-type point processes (p.p.) in a manner analogous to the generalization of univariate random variables to multivariate. The second approach reduces the multivariate case to the univariate by simply considering it as a p.p. on a more general space. These approaches will be shown to be equivalent.

Let D be a complete separable metric space (CSMS) and let $C_c(D)$ be the continuous functions with compact support on D. If Z^+ designates the non-negative integers, then let $\mathscr{N}$ be the space of Z^+-valued measures on D with the vague topology. A m.p.p. with n distinguishable types of particles may be described in either of the following ways:

(a) Let $\mathscr{N}_i = \mathscr{N}$, $i = 1,..., n$ and furnish $\mathscr{N}^n = \mathscr{N}_1 \times \cdots \times \mathscr{N}_n$ with the product topology (which will also be called vague). Then if "$\to^v$" indicates vague convergence both in $\mathscr{N}$ and in $\mathscr{N}^n$, for $N_{i_k} \in \mathscr{N}$, $N_i \in \mathscr{N}$, $i = 1,..., n$ $(N_{1_k},..., N_{n_k}) \to^v (N_1,..., N_n)$, as $k \to \infty$ iff $N_{i_k} \to^v N_i$, $i = 1,..., n$, as $k \to \infty$. Let $\mathscr{B}(\mathscr{N}^n)$ be the Borel sets of $\mathscr{N}^n$. A m.p.p. is a measurable mapping N from a probability space $(\Omega, \mathscr{F}, P)$ to $(\mathscr{N}^n, \mathscr{B}(\mathscr{N}^n))$.

The multivariate probability generating functional (PGF) is defined in the natural way on all vectors $(\phi_1,..., \phi_n)$, where $1 - \phi_i \in C_c(D)$, $0 \leqslant \phi_i \leqslant 1$, $i = 1,..., n$:

$$G(\phi_1,..., \phi_n) = E\left[\exp \sum_1^n \int_D \log \phi_i(x)\, dN_i(x)\right].$$

Similarly, the Laplace transform (LT) is defined on $(\phi_1,..., \phi_n)$ where $\phi_i \in C_c(D)$, $0 \leqslant \phi_i \leqslant 1$ in the following way:

$$L(\phi_1,..., \phi_n) = E\left[\exp - \sum_1^n \int_D \phi_i(x)\, dN_i(x)\right].$$

(b) Let $\{1,..., n\}$ be furnished with the discrete topology. Let $D' = D \times \{1,..., n\}$ be given the product topology, making D' a Polish space. Let $\mathscr{B}(D')$ be the Borel sets of D'. Let $\mathscr{N}' = \mathscr{N}(D')$ be the Z^+-valued Borel measures on D'. For $N \in \mathscr{N}'$, denote $N|_{D \times \{i\}} = N_i$, $i = 1,..., n$. N may be specified by $(N_1,..., N_n)$. (For $A \in \mathscr{B}(D')$, $N(A) = \sum_1^n N(A \cap (D \times \{i\})) = \sum_1^n N_i(A, i)$.) It is easy to see that $N_k \to^v N$ as $k \to \infty$ if and only if $N_{i_k} \to^v N_i$ as $k \to \infty$, on $D \times \{i\}$ for $i = 1,..., n$. Let $\mathscr{B}(\mathscr{N}')$ be the Borel sets of $\mathscr{N}'$. A m.p.p. is a measurable mapping N from a probability space $(\Omega, \mathscr{F}, P)$ to $(\mathscr{N}', \mathscr{B}(\mathscr{N}'))$.

The PGF of N is defined for $\phi \in C_c(D')$, $0 \leqslant \phi \leqslant 1$, by $G(\phi) = E[\exp \int_{D'} \log \phi(x)\, dN(x)]$. However, $1-\phi \in C_c(D')$ if and only if $(1-\phi)|_{D\times\{i\}} \in C_c(D \times \{i\})$. Let

$$1-\phi_i = (1-\phi)|_{D\times\{i\}}, \qquad \phi_i = \phi|_{D\times\{i\}}.$$

Thus, the PGF may be written as

$$\begin{aligned} G(\phi_1,\ldots,\phi_n) &= E\left[\exp \int_{\bigcup_1^n D\times\{i\}} \log \phi(x)\, dN(x)\right] \\ &= E\left[\exp \sum_1^n \int_{D\times\{i\}} \log \phi_i(x,i)\, dN(x,i)\right]. \end{aligned}$$

Similarly $L(\phi) = L(\phi_1,\ldots,\phi_n) = E[\exp - \sum_1^n \int_{D\times\{i\}} \phi_i(x,i)\, dN(x,i)]$, for $\phi_i \in C_c(D')$, $0 \leqslant \phi_i \leqslant 1$, $i = 1,\ldots,n$.

Therefore the two approaches yield equivalent definitions of a m.p.p. Generalizations of single-type p.p. theory are immediate in the context of definition (b), and so multitype point processes will be considered to be defined by (b) for this paper.

Let $M_{(k)}((A_1,j_1),\ldots,(A_k,j_k))$ be the kth order factorial moment of $N(A_1,j_1),\ldots,N(A_k,j_k)$, where $N(A_i,j_i) =$ (number of type j_i particles in A_i), $A_i \in \mathscr{B}(D)$, $j_i \in \{1,\ldots,n\}$, $i = 1,\ldots,k$, $k = 0, 1, 2,\ldots$. By Moyal [10], whenever it exists,

$$\begin{aligned} &M_{(k)}((A_1,j_1),\ldots,(A_k,j_k)) \\ &\quad = \lim_{\phi\to 1} \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k} G\left[\phi + \sum_1^k \lambda_i \xi_i\right]_{\lambda_1 = \cdots = \lambda_k = 0}, \end{aligned} \tag{2.1}$$

where $\xi_i = \chi(A_i,j_i)$, $i = 1,\ldots,k$. (χ is the indicator function.)

A relation analogous to (2.1) exists between $\log G$ and the factorial cumulants $C_{(k)}((A_1,j_1),\ldots,(A_k,j_k))$.

If the factorial moment (cumulant) density function of order k exists, it will be denoted by $P_k((x_1,j_1),\ldots,(x_k,j_k))$ $(Q_k(((x_1,j_1),\ldots,(x_k,j_k)))$.

A Markov m.p.p. is a Markov process N_t with state space $(\mathscr{N}', \mathscr{B}(\mathscr{N}'))$, and time homogeneous transition probabilities $P_t(\cdot \mid N)$, $N \in \mathscr{N}$. A Markov m.p.p. is multiplicative if, for all $N_1,\ldots,N_k \in \mathscr{N}$, $1 \leqslant k \leqslant \infty$, $P_t(\cdot \mid \sum_1^k N_i) = P_t(\cdot \mid N_1) * \cdots * P_t(\cdot \mid N_k)$. ($*$ denotes convolution.) In particular, if N_i is the point process consisting of a single point $(x_i,j_i) \in D'$ it follows that if $G(\cdot, t)$ is the PGF associated with N_t, then N_t is multiplicative iff

$$G(\phi, t \mid \{(x_i,j_i)_1^k\}) = \prod^k G(\phi, t \mid (x_i,j_i)).$$

For a multiplicative process, therefore, it is sufficient to consider $G(\phi, t \mid (x, i))$. The following version of the Moyal equation may be given (Moyal [10]):

$$G(\phi, t \mid (x, i)) = G^{\circ}(\phi, t \mid (x, i))$$
$$+ \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \int_0^t \int_{D^{j_1+\cdots+j_n}}$$
$$\times \prod_{l=1}^{n} \prod_{h_l=1}^{j} G(\phi, t-s \mid (y_{h_l}, l))$$
$$R^{j_1,\ldots,j_n}(y^{j_1},\ldots, y^{j_n}, s \mid (x, i))\, d(y^{j_1} \cdots y^{j_n})\, ds, \tag{2.2}$$

where $y^{j_l} = (y_1^l,\ldots, y_{j_l}^l)$.

(i) $G^{\circ}(\phi, t \mid (x, i))$ is the PGF of the probability of a transition without jumps (i.e., the total number of each type of particle does not change).

(ii) $R = \sum_{j_1} \cdots \sum_{j_n} R^{j_1,\ldots,j_n}$ defines the joint distribution of the first jump time and the consequent state. In particular, $R^{j_1,\ldots,j_n}(y^{j_1},\ldots, y^{j_n}, s \mid (x, i))$ is the probability that given one particle of type i initially, the first jump occurs at time s and results in j^l particles of type l, located at $y_1^l,\ldots, y_{j_l}^l$, respectively, for $1 \leqslant l \leqslant n$.

When it exists, the kth order factorial moment density function evaluated at $(y_1, j_1),\ldots, (y_k, j_k)$ of a Markov m.p.p. at time t given an initial type i particle at $x \in D$ is denoted by $P_k(t, (y_1, j_1),\ldots, (y_k, j_k) \mid (x, i))$. Similarly, the corresponding factorial cumulant density is denoted by $Q_k(t, (y_1, j_1),\ldots, (y_k, j_k) \mid (x, i))$.

Finally, a m.p.p. will be said to be spatially homogeneous if

$$P(N(A_1, j_1) \in H_1,\ldots, N(A_i, j_i) \in H_i)$$
$$= P(N(A_1 + x, j_1) \in H_1,\ldots, N(A_l + x, j_i) \in H_i),$$

for $A_h \in \mathscr{B}(D)$, $j_h \in \{1,\ldots, n\}$, $H_h \in \mathscr{B}(D)$, $h = 1,\ldots, i$, $i = 1, 2,\ldots$, $x \in D$.

3. Central Limit Theorems for Infinitely Divisible Point Processes

The PGF of an infinitely divisible multitype point process may be characterized as follows:

Proposition 3.1 (cf., Westcott [12, Theorem 5]). *A multivariate point process is infinitely divisible if and only if its PGF has the form*

$$G(\phi) = G(\phi_1, ..., \phi_n)$$
$$= \exp\left[\int_{\mathcal{N}'} \left(\exp\left(\sum_{i=1}^{n} \int_D \log \phi(x, i)\, N(dx, i)\right) - 1\right) \tilde{P}(dN)\right], \quad (3.1)$$

where $\tilde{P}$ is a measure on $\mathcal{N}'$ such that $\tilde{P}(\varnothing) = 0$ and for any bounded set $A \in \mathscr{B}(D)$, $\tilde{P}(N(A, i) > 0) < \infty$, for $\forall i \in \{1, ..., n\}$.

Proof. Westcott's Theorem 5 [12] expresses $G(\phi)$ in the following manner:

$$G(\phi) = \exp\left[\int_{\mathcal{N}'} \left(\exp \int_{D'} \log \phi(y)\, N(dy)\right) - 1\right] \tilde{P}(dN),$$

where $\tilde{P}(N(A) > 0) < \infty$, for all bounded sets in D'. The simplification to (3.1) is trivial, and

$$\tilde{P}(N(A) > 0) = \tilde{P}\left(\bigcup_{i=1}^{n} (N(A, i) > 0)\right) \leqslant \sum_{1}^{n} \tilde{P}(N(A, i) > 0).$$

Also, since $(A, i) \subset A$,

$$\tilde{P}(N(A, i) > 0) \leqslant \tilde{P}(N(A) > 0).$$
$$\therefore \tilde{P}(N(A, i) > 0) < \infty, i = 1, ..., n \Leftrightarrow \tilde{P}(N(A) > 0) < \infty. \quad \blacksquare$$

When $D = R^d$ (d-dimensional Euclidean space), mixing conditions for spatially homogeneous m.p.p.'s may be defined in the same way as they are for single-type point processes (Ivanoff [8]). For example, let the translation operator T_x be defined on $\mathcal{N}'$ as follows for all $x \in D$:

$$T_x(N(A)) = T_x(N(A, 1), ..., N(A, n))$$
$$= (N(A + x, 1), ..., N(A + x, n)).$$

Then $N(\cdot)$ is mixing if

$$\lim_{x \to \infty} P(A T_x B) \to P(A) P(B) \qquad \text{for all} \quad A, B \in \mathscr{B}(\mathcal{N}').$$

A second concept of mixing (B-mixing) is due to Brillinger [3]. Let $N \in \mathcal{N}'$ be spatially homogeneous. N satisfies the B-mixing condition of order l if, for all $k \leqslant l$, Q_k exists and $\int_D \cdots \int_D |Q_k((x_1, j_1), ..., (x_k, j_k))|\, dx_1 \cdots dx_{k-1} < \infty$. Define

$$\int_D \cdots \int_D |Q_k((x_1, j_1), ..., (x_k, j_k))|\, dx_1 \cdots dx_{k-1}$$
$$= \xi_k(j_1, ..., j_k). \quad (3.2)$$

THEOREM 3.1. *Let N be an infinitely divisible spatially homogeneous* m.p.p. *defined on $D = R^d$ (d-dimensional Euclidean space) which is B-mixing of order 3. Let $N_K(\cdot)$ be the renormalization of N defined by*

$$\int_{D'} \phi(y)\, N_k(dy) = \int_{D'} \phi\left(\frac{y}{K}\right) N(dy), \qquad \phi \in C_c(D'),$$

where $\phi(y/K) = \phi((x/K, i))$, if $y = (x, i)$.

If

$$M_K(\cdot) = \frac{N_K(\cdot) - EN_K(\cdot)}{K^{d/2}},$$

then as $K \to \infty$, $M_K(\cdot)$ converges in law to the generalized n-variate Gaussian random field on D', with covariance kernel

$$\Gamma((x, i), (y, j)) = \xi_2(i, j)\, \delta(x - y) + \xi_1(i)\, \delta(i - j)\, \delta(x - y).$$

Proof. It will be shown that the sequence of Laplace transforms (L_K) of (M_K) satisfies

$$\begin{aligned} \ln L_K(\phi) &= \ln L_K(\phi_1, ..., \phi_n) \\ &\to \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\delta(i-j)\, \xi_1(i) + \xi_2(i, j)] \\ &\quad \times \int_D \phi_i(x, i)\, \phi_j(x, j)\, dx \end{aligned} \tag{3.3}$$

as $K \to \infty$.

First, it is noted that the LT of N has the form

$$\begin{aligned} L(\phi) &= L(\phi_1, ..., \phi_n) = G(e^{-\phi_1}, ..., e^{-\phi_n}) \\ &= \exp\left[\int_{\mathcal{N}'} \left(\exp\left(-\sum_{i=1}^{n} \int_D \phi_i(x, i)\, N(dx, i)\right) - 1\right) \tilde{P}(dN)\right] \\ &= \exp\left[\int_{\mathcal{N}'} \left(\exp\left(-\int_{D'} \phi(y)\, N(dy)\right) - 1\right) \tilde{P}(dN)\right], \end{aligned}$$

where $y \in D \times \{1, ..., n\}$, and $\phi_i = \phi|_{D \times i}$.

Denote the jth order joint cumulant of $(N(\phi^1), ..., N(\phi^j))$, where $\phi^l = (\phi_1^l, ..., \phi_n^l)$, and $N(\phi^l) = \int_{D'} \phi^l(y)\, N(dy)$, $l = 1, ..., j$, by $C_j(\phi^1, ..., \phi^j)$. When it exists,

$$C_j(\phi^1,\dots,\phi^j)=(-1)^j\frac{\partial^j}{\partial\lambda_1\cdots\partial\lambda_j}\ln L\left(\sum_{l=1}^{j}\lambda_l\phi^l\right)_{\lambda_1=\cdots=\lambda_j=0}$$

$$=\int_{\mathscr{N}'}\exp(-N(\lambda_1\phi^1)-\cdots-N(\lambda_j\phi^j))N(\phi^1)\cdots N(\phi^j)\tilde{P}dN|_{\lambda_1=\cdots=\lambda_j=0}$$

$$=\int_{\mathscr{N}'}\left[\int_{D'}\phi^1(x)\,N(dx)\right]\cdots\left[\int_{D'}\phi^j(x)\,N(dx)\right]\tilde{P}\,dN. \tag{3.4}$$

If a jth order cumulant kernel exists, it will be designated by $C_j(y_1,\dots,y_j)$, $y_i\in D'$, $i=1,\dots,j$. Then

$$C_j(\phi^1,\dots,\phi^j)=\int_{D'}\cdots\int_{D'}\phi^1(y_1)\cdots\phi^j(y_j)\,C_j(y_1,\dots,y_j)\,dy_1\cdots dy_j. \tag{3.5}$$

Clearly, $C_1\equiv Q_1$.

We have

$$L_K(\phi)=E\left(\exp\left[-\int_{D'}\phi(y)\,M_K(dy)\right]\right)$$

$$=E\left(\exp\left(-\frac{1}{K^{d/2}}\int_{D'}\phi\left(\frac{y}{K}\right)N(dy)\right.\right.$$

$$\left.\left.+\frac{1}{K^{d/2}}\int_{D'}Q_1(y)\phi\left(\frac{y}{K}\right)dy\right)\right),$$

where, if $y=(x,i)$, then $(y/K)=(x/K,i)$. If $\phi_K(y)=\phi(y/K)$, then

$$\ln L_K(\phi)=\left[\frac{1}{K^{d/2}}\int_{D'}Q_1(y)\phi\left(\frac{y}{K}\right)dy\right]+\ln L(\phi_K/K^{d/2})$$

$$=\frac{1}{K^{d/2}}\int_{D'}Q_1(y)\phi\left(\frac{y}{K}\right)dy$$

$$+\int_{\mathscr{N}'}\left[\exp\left(-\frac{1}{K^{d/2}}\int_{D'}\phi\left(\frac{y}{K}\right)N(dy)\right)-1\right]\tilde{P}(dN)$$

$$=\frac{1}{K^{d/2}}\int_{D'}Q_1(y)\phi\left(\frac{y}{K}\right)dy$$

$$+\int_{\mathscr{N}'}\left[1-\frac{1}{K^{d/2}}\int_{D'}\phi\left(\frac{y}{K}\right)N(dy)\right.$$

$$\left.+\frac{1}{2K^d}\left\{\int_{D'}\phi\left(\frac{y}{K}\right)N(dy)\right\}^2+\varepsilon_K-1\right]\tilde{P}(dN),$$

where

$$|\varepsilon_K| \leqslant \frac{1}{6K^{3d/2}} \left\{ \int_{D'} \phi\left(\frac{y}{K}\right) N(dy) \right\}^3.$$

But

$$\begin{aligned} \int_{\mathcal{N}'} \frac{1}{K^{d/2}} \int_{D'} \phi\left(\frac{y}{K}\right) N(dy)\, \tilde{P}(dN) & \\ = C_1(\phi_K/K^{d/2}) & \qquad \text{(by 3.4)} \\ = \frac{1}{K^{d/2}} \int_{D'} Q_1(y) \phi\left(\frac{y}{K}\right) dy & \qquad \text{(by 3.5).} \end{aligned}$$

Now,

$$\begin{aligned} &\int_{\mathcal{N}'} \frac{1}{K^d} \left\{ \int_{D'} \phi\left(\frac{y}{K}\right) N(dy) \right\}^2 \tilde{P}(dN) \\ &= C_2(\phi_K/K^{d/2}, \phi_K/K^{d/2}) \\ &= \frac{1}{K^d} \int_{D'} \int_{D'} \phi\left(\frac{y_1}{K}\right) \phi\left(\frac{y_2}{K}\right) C_2(y_1, y_2)\, dy_1\, dy_2 \\ &= \frac{1}{K^d} \int_{D'} \int_{D'} \phi\left(\frac{y_1}{K}\right) \phi\left(\frac{y_2}{K}\right) (Q_2(y_1, y_2) \\ &\quad + \delta(y_1 - y_2)\, Q_1(y_1))\, dy_1\, dy_2 \\ &= K^d \int_{D'} \int_{D'} \phi(y_1)\, \phi(y_2) (Q_2(Ky_1, Ky_2) \\ &\quad + \delta \frac{(y_1 - y_2)}{K^d} Q_1(Ky_1))\, dy_1\, dy_2, \end{aligned}$$

where, if $y = (x, i)$, $Ky = (Kx, i)$. Finally,

$$\begin{aligned} \left| \int_{\mathcal{N}'} \varepsilon_K \tilde{P}\, dN \right| &\leqslant \left| \frac{1}{6K^{3d/2}} \int_{D'} \int_{D'} \int_{D'} \phi\left(\frac{y_1}{K}\right) \phi\left(\frac{y_2}{K}\right) \phi\left(\frac{y_3}{K}\right) \right. \\ &\qquad \left. \times C_3(y_1, y_2, y_3)\, dy_1\, dy_2\, dy_3 \right| \\ &\leqslant \frac{1}{6} \left| \left[\frac{1}{K^{3d/2}} \int_{KA} \int_{D'} \int_{D'} Q_3(y_1, y_2, y_3)\, dy_1\, dy_2\, dy_3 \right. \right. \\ &\qquad + \sum_{1 \leqslant i < j \leqslant 3} \frac{1}{K^{3d/2}} \int_{KA} \int_{D'} Q_2(y_i, y_j)\, dy_i\, dy_j \\ &\qquad \left. \left. + \frac{1}{K^{3d/2}} \int_{KA} Q_1(y)\, dy \right] \right|, \end{aligned}$$

where $A = \operatorname{supp} \phi$, and $KA = \{Ky : y \in A\}$. Therefore,

$$\left|\int_{\cdot\, r'} \varepsilon_K \tilde{P}\, dN\right| \leqslant \frac{|A|}{6K^{d/2}} \left[\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{j_3=1}^{n} \xi_3(j_1, j_2, j_3) + 3 \sum_{i_1=1}^{n} \sum_{j_2=1}^{n} \xi_2(j_1, j_2) + \sum_{j=1}^{n} \xi_1(j)\right],$$

where $|A| = \sum_{j=1}^{n} |(A, j)|$. Finally,

$$\begin{aligned}\ln L_K(\phi) = \frac{K^d}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_D \int_D & \phi_i(x_1, i)\, \phi_j(x_2, j) \\ & \times Q_2((Kx_1, i), (Kx_2, j))\, dx_1\, dx_2 \\ & + \frac{1}{2} \sum_{i=1}^{n} \int_D \phi_i^2(x, i)\, Q_1(Kx, i)\, dx \\ & + O(K^{-d/2}).\end{aligned}$$

But if $f(x, y) = f(x - y)$, and $\int_D f(\omega)\, d\omega = F < \infty$, then $K^d f(K\omega) \to F\delta(\omega)$ in the generalized sense. Therefore, as $K \to \infty$

$$\begin{aligned}\ln L_K(\phi) \to \frac{1}{2} \sum_{i=1} \sum_{j=1} & [\xi_2(i, j) + \delta(i - j)\, \xi_1(i)] \\ & \times \int_D \phi_i(x, i)\, \phi_j(x, j)\, dx.\end{aligned}$$

This proves (3.3). ∎

Theorem 3.1 requires only that the finite dimensional distributions of M_K converge in law to the appropriate multivariate Gaussian distribution. However, for $t = (t_1, \ldots, t_d)$, $0 \leqslant t_i < \infty$, $i = 1, \ldots, d$, define

$$X_K(t_1, \ldots, t_d) = M_K([0, t_1] \times \cdots \times [0, t_d]). \tag{3.6}$$

By definition $X_K \in D[[0, \infty)^d, R^n]$, the Skorokhod function space $[0, \infty)^d \to R^n$ consisting of functions "continuous from above with limits from below" in the sense of Bickel and Wichura [2] and Ivanoff [9]. An additional tightness condition is required to prove weak convergence of the functions $X_K(\cdot)$ in the Skorokhod topology.

THEOREM 3.2. *Assume that all the hypotheses of Theorem* 3.1 *hold. In addition, let N be B-mixing of order* 4. *If* $X_K(t)$ *is defined by* (3.6), *then as* $K \to \infty$, $X_K(\cdot)$ *converges weakly in the Skorokhod topology to* $G(\cdot)$, *the* *multivariate Gaussian process with independent increments*; $G(t)=0$ *and* $G(t)$ *has covariance matrix* $\Gamma \cdot (t_1 \cdot \ldots \cdot t_d)$, *where* $t=(t_1,\ldots,t_d)$, *and*

$$\Gamma = \begin{bmatrix} \xi_1(1)+\xi_2(1,1) & \xi_2(1,2) & \cdots & \xi_2(1,n) \\ \xi_2(1,2) & \xi_1(2)+\xi_2(2,2) & & \xi_2(2,n) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_2(1,n) & \xi_2(2,n) & \cdots & \xi_1(n)+\xi_2(n,n) \end{bmatrix}.$$

Proof. The convergence of the finite dimensional distributions is proven in Theorem 3.1. It remains to show that the sequence $X_K(\cdot)$ is tight. By Theorem 5.1 of Ivanoff [9], it is sufficient to show that for all $T>0$, $X_K(\cdot)=(X_K^1(\cdot),\ldots,X_K^n(\cdot))$ satisfies

$$\lim_{\delta\to 0}\limsup_{K\to\infty} P[\omega_\delta^T(X_K^i)\geqslant\varepsilon]=0,$$

for all $\varepsilon>0$, where

$$\omega_\delta^T(X^i)=\sup_{\substack{0\leqslant s_j,t_j\leqslant T\\ j=1,\ldots,d\\ |s_j-t_j|\leqslant\delta}} |X^i(t)-X^i(s)|.$$

Thus, the point process associated with each type of point may be considered separately, which reduces the tightness problem to the 1-dimensional case. The proof is identical to the tightness proof in Theorem 7.1 and Corollary 7.1 of Ivanoff [8], and will not be repeated here. ∎

4. THE MULTITYPE BRANCHING DIFFUSION (MBD)

The simple MBD is a multiplicative Markov m.p.p. in which particles independently undergo migration in D and branching. It will be assumed that $D=\mathscr{R}^d$. The process is defined by:

(a) The migration process: All particles undergo a time-homogeneous migration specified by a transition density $p(t,a,b)$ where a is the initial location of the particle. All particles obey the same law, regardless of the type of particle. It will be assumed that $p(s,x,y)=p(s,y,x)=p(s,x-y,0)$. Define the operator T_t by $T_tf(x)=\int_D p(t,x,y)f(y)\,dy$. By the Markovian nature of the process, $\int_D p(t,x,y)p(s,y,z)\,dy=p(t+s,x,z)$, and so the operators T_t form a semigroup.

(b) The branching rates: Particles of type i are assumed to have an exponential lifetime with mean V_i^{-1}, $i=1,\ldots,n$. V_i is the ith branching rate.

(c) The offspring distribution: When a type i particle branches, it produces $j_1,\ldots, j_n$ offspring of type 1,..., type n, respectively, with probability $p^i(j_1,\ldots, j_n)$. All offspring are born at the location in D where the parent branched. Let $P^i(t_1,\ldots, t_n) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} p^i(j_1,\ldots, j_n)\, t_1^{j_1} \cdots t_n^{j_n}$ be the PGF of the offspring distribution of a type i particle.

It is assumed that the process begins initially with one type i particle at $x \in D$. The resulting m.p.p. at time t is denoted by $N(\,\cdot\,, t \mid (x, i))$.

Using these hypotheses, the Moyal equation (2.2) for the PGF $G(\,\cdot\,, t \mid (x, i))$ becomes

$$\begin{aligned}
&G(\phi, t \mid (x, i))\\
&\quad = e^{-V_i t} \int_D \phi_i(y)\, p(t, x, y)\, dy\\
&\qquad + \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} p^i(j_1,\ldots, j_n)\\
&\qquad \times \int_0^t \int_D \prod_{l=1}^{n} G(\phi, t-s \mid (y, l))^{j_l} V_i\, e^{-V_i s} p(s, x, y)\, dy\, ds\\
&\quad = e^{-V_i t} \int_D \phi_i(y)\, p(t, x, y)\, dy\\
&\qquad + \int_0^t \int_D V_i e^{-V_i s} p(s, x, y)\\
&\qquad \times P^i(G(\phi, t-s \mid (y, 1)),\ldots, G(\phi, t-s \mid (y, n)))\, dy\, ds.
\end{aligned} \tag{4.1}$$

Using (2.1), equations for the factorial moments can be obtained from (4.1) by differentiation.

THEOREM 4.1. *Let $E^i(j)$ be the expected number of type j offspring produced by the branch of a type i particle in a MBD. If $E^i(j) < \infty$, $0 \leqslant i$, $j \leqslant n$, then the first factorial moment density exists, and $P_1(t, (y, j) \mid (x, i)) = C_{ij}(t)\, p(t, x, y)$. $C_{ij}(t)$ is the ijth entry of the matrix $C(t) = e^{Mt}$, where*

$$M = \begin{bmatrix} V_1(E^1(1)-1) & V_1 E^1(2) & \cdots & V_1 E^1(n)\\ V_2 E^2(1) & V_2(E^2(2)-1) & \cdots & V_2 E^2(n)\\ \vdots & \vdots & \ddots & \vdots\\ V_n E^n(1) & V_n E^n(2) & \cdots & V_n(E^n(n)-1) \end{bmatrix}.$$

Proof. Differentiating (4.1), one obtains the following renewal-type equation for $P_1(t, (y, j) \mid (x, i))$:

$$P_1(t, (y, j) \mid (x, i))$$
$$= \int_0^t \int_D V_i e^{-V_i u} p(u, x, z) \sum_{k=1}^n P_1(t-u, (y, j) \mid (z, k)) E^i(k)\, dz\, du$$
$$+ \delta(i-j)\, e^{-V_i t} p(t, x, y). \tag{4.2}$$

Taking Laplace transforms of both sides in (4.2),

$$L_\lambda(P_1(\cdot, (y, j) \mid (x, i)))$$
$$= \sum_{k=1}^n V_i E^i(k) \int_0^\infty e^{-\lambda u} e^{-V_i u} \int_D p(u, x, z) L_\lambda(P_1(*, (y, j) \mid (z, k)))\, dz\, du$$
$$+ \delta(i-j) \int_0^\infty e^{-\lambda t} e^{-V_i t} p(t, x, y)\, dt. \tag{4.3}$$

Let $T_t^i f(x) = \int_D e^{-V_i t} p(t, x, y) f(y)\, dy$. If A is the infinitesimal generator of T_t, then $A - V_i I$ is the infinitesimal generator of T_t^i. Let R_λ and R_λ^i be the resolvent operators of T_t and T_t^i, respectively. Equation (4.3) can be written as

$$L_\lambda(P_1(\cdot, (y, j) \mid (x, i)))$$
$$= \sum_{k=1}^n V_i E^i(k) R_\lambda^i L_\lambda(P_1(\cdot, (y, j) \mid (x, k))) + \delta(i-j) R_\lambda^i(\delta(y)). \tag{4.4}$$

$$(\lambda - (A - V_i)) L_\lambda(P_1(\cdot, (y, j) \mid (x, i)))$$
$$= \sum_{k=1}^n V_i E^i(k) L_\lambda(P_1(\cdot, (y, j) \mid (x, k))) + \delta(i-j)\, \delta(y) \tag{4.5}$$

$$\Rightarrow (\lambda - (A - V_i(1 - E^i(i)))) L_\lambda P_1(\cdot, (y, j) \mid (x, i))$$
$$= \sum_{k \neq i} V_i E^i(k) L_\lambda(P_1(\cdot, (y, j) \mid (x, k))) + \delta(i-j)\, \delta(y).$$

Let Q_λ^i be the Laplace transform of the semigroup S_t^i, where $S_t^i f(x) = \int_D e^{-V_i(1-E^i(i))t} p(t, x, y) f(y)\, dy$. The infinitesimal generator of S_t^i is $A - V_i(1 - E^i(i))$. Therefore,

$$L_\lambda(P_1(\cdot, (y, j) \mid (x, i)))$$
$$= \sum_{k \neq i} V_i E^i(k) Q_\lambda^i L_\lambda P_1(\cdot, (y, j) \mid (x, k)) + Q_\lambda^i \delta(i-j)\, \delta(y).$$

$$\Rightarrow \int_0^\infty e^{-\lambda t} P_1(t, (y, j) \mid (x, i))\, dt$$

$$= \sum_{k \neq i} V_i E^i(k) \int_0^\infty e^{-\lambda s} e^{-V_i(1-E^i(i))s} \int_D p(s, x, z)$$

$$\times \int_0^\infty e^{-\lambda u} P_1(u, (y, j) \mid (z, k))\, du\, dz\, ds$$

$$+ \delta(i-j) \int_0^\infty e^{-\lambda t} e^{-V_i(1-E^i(i))t} p(t, x, y)\, dt$$

$$= \sum_{k \neq i} V_i E^i(k) \int_0^\infty e^{-\lambda t} \left[\int_0^t \int_D e^{-V_i(1-E^i(i))s} p(s, x, z) \right.$$

$$\left. \times P_1(t-s, (y, j) \mid (z, k))\, dz\, ds \right] dt$$

$$+ \delta(i-j) \int_0^\infty e^{-\lambda t} e^{-V_i(1-E^i(i))t} p(t, x, y)\, dt.$$

Inverting Laplace transforms,

$$P_1(t, (y, j) \mid (x, i)) = \sum_{k \neq i} V_i E^i(k) \int_0^t \int_D e^{-V_i(1-E^i(i))s} p(s, x, z) \times P_1(t-s, (y, j) \mid (z, k))\, dz\, ds + e^{-V_i(1-E^i(i))t} p(t, x, y)\, \delta(i-j). \tag{4.6}$$

Equation (4.6) gives a system of n equations. It can be seen by iteration that the solution is of the form $P_1(t, (y, j) \mid (x, i)) = C_{ij}(t)\, p(t, x, y)$. Substituting in (4.6), it is seen that $C_{ij}(t)$ satisfies

$$C_{ij}(t) = \sum_{k \neq i} V_i E^i(k) \int_0^t e^{-V_i(1-E^i(i))(t-u)} C_{kj}(u)\, du + \delta(i-j)\, e^{-V_i(1-E^i(i))t}. \tag{4.7}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t} C_{ij}(t) = -\sum_{k \neq i} V_i E^i(k) \int_0^t V_i(1-E^i(i))\, e^{-V_i(1-E^i(i))(t-u)} C_{kj}(u)\, du \\ \qquad + \sum_{k \neq i} V_i E^i(k)\, C_{kj}(t) - \delta(i-j)\, V_i(1-E^i(i))\, e^{-V_i(1-E^i(i))t} \\ C_{ij}(0) = \delta(i-j) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t} C_{ij}(t) = V_i(E^i(i) - 1)\, C_{ij}(t) + \sum\limits_{k \neq j} V_i E^i(k)\, C_{kj}(t) \\ C_{ij}(0) = \delta(i-j) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial t} C(t) = MC(t), \text{ where } M \text{ is defined in the statement of the theorem} \\ C(0) = I \end{cases}$$

$$\Rightarrow \qquad C(t) = e^{Mt}. \quad \blacksquare$$

Let $E^i(j_1, \ldots, j_k)$ be the kth order joint factorial moment of the numbers of type $j_1, j_2, \ldots, j_k$ particles resulting from the branch of a type i particle. If $E^i(j_1, \ldots, j_k) < \infty$ for all $1 \leqslant j_l \leqslant n$, $l = 1, \ldots, k$, $i = 1, \ldots, n$, then the kth order factorial moment density of $N(\cdot, t \mid (x, i))$ exists. In particular, the following theorem will be needed.

THEOREM 4.2. *If* $E^i(j_1, j_2) < \infty$, *for all* $1 \leqslant j_l \leqslant n$, $l = 1, 2$, $1 \leqslant i \leqslant n$, *then the second factorial moment density of the MBD exists and*

$$\begin{aligned} &P_2(t, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &\quad = \sum_{l=1}^{n} V_l \int_0^t \int_D P_1(t-s, (w, l) \mid (x, i)) \\ &\quad \times f_2(s, (y_1, j_1), (y_2, j_2) \mid (w, l))\, dw\, ds, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} &f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, l)) \\ &\quad = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} E^l(k_1, k_2)\, P_1(u, (y_1, j_1) \mid (z, k_1)) \\ &\quad \times P_1(u, (y_2, j_2) \mid (z, k_2)). \end{aligned}$$

Proof. Using (2.1) and (4.1), the following equation for P_2 may be obtained:

$$\begin{aligned} &P_2(t, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &\quad = \int_0^t \int_D V_i e^{-V_i u} p(u, x, y)[f_2(t-u, (y_1, j_1), (y_2, j_2) \mid (z, i)) \\ &\quad + \sum_{k=1}^{n} E^i(k)\, P_2(t-u, (y_1, j_1), (y_2, j_2) \mid (z, k))]\, dz\, du. \end{aligned} \tag{4.9}$$

Taking Laplace transforms in (4.9), and using the same notation as in the proof of Theorem 4.1, the following equation is obtained:

$$\begin{aligned} L_\lambda P_2&(\cdot, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &= V_i R_\lambda^i \Big[L_\lambda f_2(\cdot, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &\quad + \sum_{k=1}^{n} E^i(k) L_\lambda P_2(\cdot, (y_1, j_1), (y_2, j_2) \mid (x, k)) \Big]. \end{aligned} \tag{4.10}$$

$$\begin{aligned} \Rightarrow (\lambda - (A - V_i(1 - E^i(i))) L_\lambda P_2&(\cdot, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &= V_i L_\lambda f_2(\cdot, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &\quad + V_i \sum_{k \neq i} E^i(k) L_\lambda P_2(\cdot, (y_1, j_1), (y_2, j_2) \mid (x, k)) \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^\infty & e^{-\lambda t} P_2(t, (y_1, j_1), (y_2, j_2) \mid (x, i))\, dt \\ &= V_i \int_0^\infty e^{-\lambda s} e^{-V_i(1-E^i(i))s} \int_D p(s, x, z) \\ &\quad \times \Big[\int_0^\infty e^{-\lambda u} f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, i))\, du \\ &\quad + \sum_{k \neq i}^{n} E^i(k) \int_0^\infty e^{-\lambda u} P_2(u, (y_1, j_1), (y_2, j_2) \mid (z, k))\, du \Big] dz\, ds \end{aligned}$$

$$\begin{aligned} \Rightarrow P_2&(t, (y_1, j_1), (y_2, j_2) \mid (x, i)) \\ &= \int_0^t \int_D V_i e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z) \\ &\quad \times f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, i))\, dz\, du \\ &\quad + \sum_{k \neq i}^{n} E^i(k) \int_0^t \int_D V_i e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z) \\ &\quad \times P_2(u, (y_1, j_1), (y_2, j_2) \mid (z, k))\, dz\, du. \end{aligned} \tag{4.11}$$

Equation (4.11) gives a system of n equations for P_2. Inspection of the univariate solution (see, for example, Ivanoff [6]) suggests a solution of the form (4.8). Substituting (4.8) in the right-hand side of (4.11), one obtains

$$RHS = V_i \int_0^t \int_D e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z)$$
$$\times f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, i))\, dz\, du$$
$$+ \sum_{k \neq i}^{n} E^i(k) \int_0^t \int_D \int_0^u \int_D V_i e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z)$$
$$\times \sum_{l=1}^{n} V_l C_{kl}(u-s)\, p(u-s, z, w)$$
$$\times f_2(s, (y_1, j_1), (y_2, j_2) \mid (w, l))\, dw\, ds\, dz\, du$$
$$= V_i \int_0^t \int_D e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z)$$
$$\times f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, i))\, dz\, du$$
$$+ \sum_{k \neq i}^{n} E^i(k) \int_0^t \int_D \int_0^u p(t-s, x, w)\, V_i e^{-V_i(1-E^i(i))(t-u)}$$
$$\times \sum_{l=1}^{n} V_l C_{kl}(u-s) f_2(s, (y_1, j_1), (y_2, j_2) \mid (w, l))\, ds\, dw\, du$$
$$= V_i \int_0^t \int_D e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z) f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, i))\, dz\, du$$
$$+ \sum_{l=1}^{n} V_l \int_0^t \int_D p(t-s, x, w) f_2(s, (y_1, j_1), (y_2, j_2) \mid (w, l))$$
$$\times \sum_{k \neq i}^{n} V_i E^i(k) \int_0^{t-s} e^{-V_i(1-E^i(i))(t-s-u)} C_{kl}(u)\, du\, dw\, ds$$
$$= V_i \int_0^t \int_D e^{-V_i(1-E^i(i))(t-u)} p(t-u, x, z)$$
$$\times f_2(u, (y_1, j_1), (y_2, j_2) \mid (z, i))\, dz\, du$$
$$+ \sum_{l=1}^{n} V_l \int_0^t \int_D p(t-s, x, w) f_2(s, (y_1, j_1), (y_2, j_2) \mid (w, l))$$
$$\times \left[C_{il}(t-s) - \delta(i-l)\, e^{-V_i(1-E^i(i))(t-s)}\right] dw\, ds$$

(This follows from (4.7).)

$$= \sum_{l=1}^{n} V_l \int_0^t \int_D P_1(t-s, (w, l) \mid (x, i))$$
$$\times f_2(s, (y_1, j_1), (y_2, j_2) \mid (w, l))\, dw\, ds = LHS.$$

The uniqueness of the solution to (4.11) is easily verified by the fact that the integrand on the right-hand side satisfies a Lipschitz condition. ∎

It is straightforward, but tedious, to show that in general

$$P_k(t, (y_1, j_1),..., (y_k, j_k) \mid (x, i))$$
$$= \int_0^t \int_D \Big\{ V_i e^{-V_i u} p(u, x, z) f_k(t-u, (y_1, j_1),..., (y_k, j_k) \mid (z, i))$$
$$+ \sum_{j=1}^{n} E^i(j) P_k(t-u, (y_1, j_1),..., (y_k, j_k) \mid (z, j)) \Big\} \, dy \, du, \quad (4.12)$$

where f_k is a function involving lower-order joint factorial moment densities, and the factorial moments $E^i(h_1,..., h_l)$, where $l \leqslant k$, $h_r \in (j_1,..., j_k)$, $r = 1,..., l$.

Equation (4.12) may be solved by the method of Theorem 4.2, to give the following equation:

$$P_k(t, (y_1, j_1),..., (y_k, j_k) \mid (x, i))$$
$$= \sum_{l=1}^{n} V_l \int_0^t \int_D P_1(t-u, (z, l) \mid (x, i))$$
$$\times f_k(u, (y_1, j_1),..., (y_k, j_k) \mid (z, l)) \, dz \, du. \quad (4.13)$$

From (4.13) it is clear that P_k exists if and only if the joint moments of order $\leqslant k$ of the branching process exist. In particular, if all moments up to and including the third exist for the branching process, then it may be shown that

$$P_3(t, (y_1, j_1), (y_2, j_2), (y_3, j_3) \mid (x, i))$$
$$= \sum_{l=1}^{n} V_l \int_0^t \int_D P_1(t-u, (z, l) \mid (x, i))$$
$$\times f_3(u, (y_1, j_1), (y_2, j_2), (y_3, j_3) \mid (z, l)) \, dz \, du, \quad (4.14)$$

where

$$f_3(u, (y_1, j_1), (y_2, j_2), (y_3, j_3) \mid (z, l))$$
$$= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} E^l(k_1, k_2, k_3) P_1(u, (y_1, j_1) \mid (z, k_1))$$
$$\times P_1(u, (y_2, j_2) \mid (z, k_2)) P_1(u, (y_3, j_3) \mid (z, k_3))$$
$$+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} E^l(k_1, k_2)[P_1(u, (y_1, j_1) \mid (z, k_1))$$

$$\times P_2(u, (y_2, j_2), (y_3, j_3) \mid (z, k_2))$$
$$+ P_1(u, (y_2, j_2) \mid (z, k_1)) P_2(u, (y_1, j_1), (y_3, j_3) \mid (z, k_2))$$
$$+ P_1(u, (y_3, j_3) \mid (z, k_1)) P_2(u, (y_1, j_1), (y_2, j_2) \mid (z, k_2))].$$

If all moments up to and including the fourth exist for the branching process, then the fourth factorial moment density is given by

$$P_4(t, (y_1, j_1), (y_2, j_2), (y_3, j_3), (y_4, j_4) \mid (x, i))$$
$$= \sum_{l=1}^{n} V_l \int_0^t \int_D P_1(t-u, (z, l) \mid (x, i))$$
$$\times f_4(u, (y_1, j_1), \ldots, (y_4, j_4) \mid (z, l))\, dz\, du, \tag{4.15}$$

where

$$f_4(u, (y_1, j_1), (y_2, j_2), (y_3, j_3), (y_4, j_4) \mid (z, l))$$
$$= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} \sum_{k_4=1}^{n} E^l(k_1, k_2, k_3, k_4)$$
$$\times P_1(u, (y_1, j_1) \mid (z, k_1)) P_1(u, (y_2, j_2) \mid (z, k_2))$$
$$\times P_1(u, (y_3, j_3) \mid (z, k_3)) P_1(u, (y_4, j_4) \mid (z, k_4))$$
$$+ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} E^l(k_1, k_2, k_3)$$
$$\times [P_1(u, (y_1, j_1) \mid (z, k_1)) P_1(u, (y_2, j_2) \mid (z, k_2))$$
$$\times P_2(u, (y_3, j_3), (y_4, j_4) \mid (z, k_3))$$
$$+ P_2(u, (y_1, j_1) \mid (z, k_1)) P_1(u, (y_3, j_3) \mid (z, k_2))$$
$$\times P_2(u, (y_2, j_2), (y_4, j_4) \mid (z, k_3))$$
$$+ P_1(u, (y_1, j_1) \mid (z, k_1)) P_1(u, (y_4, j_4) \mid (z, k_2))$$
$$\times P_2(u, (y_2, j_2), (y_3, j_3) \mid (z, k_3))$$
$$+ P_1(u, (y_2, j_2) \mid (z, k_1)) P_1(u, (y_3, j_3) \mid (z, k_2))$$
$$\times P_2(u, (y_1, j_1), (y_4, j_4) \mid (z, k_3))$$
$$+ P_1(u, (y_2, j_2) \mid (z, k_1)) P_1(u, (y_4, j_4) \mid (z, k_2))$$
$$\times P_2(u, (y_1, j_1), (y_3, j_3) \mid (z, k_3))$$
$$+ P_1(u, (y_3, j_3) \mid (z, k_1)) P_1(u, (y_4, j_4) \mid (z, k_2))$$
$$\times P_2(u, (y_1, j_1), (y_2, j_2) \mid (z, k_3))]$$

$$+\sum_{k_1=1}^{n}\sum_{k_2=1}^{n} E^l(k_1,k_2)[P_2(u,(y_1,j_1),(y_2,j_2)\mid(z,k_1))$$
$$\times P_2(u,(y_3,j_3),(y_4,j_4)\mid(z,k_2))$$
$$+P_2(u,(y_1,j_1),(y_3,j_3)\mid(z,k_1))\,P_2(u,(y_2,j_2),(y_4,j_4)\mid(z,k_2))$$
$$+P_2(u,(y_1,j_1),(y_4,j_4)\mid(z,k_1))\,P_2(u,(y_2,j_2),(y_3,j_3)\mid(z,k_2))$$
$$+P_1(u,(y_1,j_1)\mid(z,k_1))\,P_3(u,(y_2,j_2),(y_3,j_3),(y_4,j_4)\mid(z,k_2))$$
$$+P_1(u,(y_2,j_2)\mid(z,k_1))\,P_3(u,(y_1,j_1),(y_3,j_3),(y_4,j_4)\mid(z,k_2))$$
$$+P_1(u,(y_3,j_3)\mid(z,k_1))\,P_3(u,(y_1,j_1),(y_2,j_2),(y_4,j_4)\mid(z,k_2))$$
$$+P_1(u,(y_4,j_4)\mid(z,k_1))\,P_3(u,(y_1,j_1),(y_2,j_2),(y_3,j_3)\mid(z,k_2))].$$

Consider now the limiting behaviour as $t\to\infty$ of $E(N((A,j),t\mid(x,i))$ and $\mathrm{Cov}(N((A,j_1),t\mid(x,i)),N((A,j_2),t\mid(x,i)))$ for a bounded set $A\in\mathscr{B}(D)$, in the special case where $D=R^d$, the migration process is Brownian motion, and the branching process is positive regular. Let λ be the (real) maximal eigenvalue of M. The process is subcritical if $\lambda<0$, critical if $\lambda=0$, and supercritical if $\lambda>0$. Let $\mathbf{u}$ and $\mathbf{v}$ be right and left eigenvectors of $C(t)$ with eigenvalue $e^{\lambda t}$ such that $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{1}=1$. By the Frobenius theorem, $\lim_{t\to\infty}C(t)\,e^{-\lambda t}=((u_iv_j))$. It follows that

$$\lim_{t\to\infty} E(N(A,j),t\mid(x,i))$$
$$=\lim_{t\to\infty}\frac{u_iv_je^{\lambda t}}{(2\pi t)^{d/2}}\int_A \exp[-|x-y|^2/2t]\,dy=\lim_{t\to\infty}\frac{u_iv_je^{\lambda t}}{(2\pi t)^{d/2}}|A|; \tag{4.16}$$

$$\lim_{t\to\infty}\mathrm{Cov}(N((A,j_1),t\mid(x,i)),N((A,j_2),t\mid(x,i)))$$
$$=\lim_{t\to\infty}\Big[\int_A\int_A P_2(t,(y_1,j_1),(y_2,j_2)\mid(x,i))\,dy_1\,dy_2$$
$$+\delta(j_1-j_2)\,E(N((A,j_1),t\mid(x,i))$$
$$-E(N((A,j_1),t\mid(x,i)))\,E(N((A,j_2),t\mid(x,i)))\Big]; \tag{4.17}$$

$$\int_A\int_A P_2(t,(y_1,j_1),(y_2,j_2)\mid(x,i))\,dy_1\,dy_2$$
$$=\sum_{l=1}^{n}V_l\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}E^l(k_1,k_2)\int_0^t C_{il}(t-s)C_{k_1j_1}(s)\,C_{k_2j_2}(s)$$
$$\times\frac{1}{(2\pi(t-s))^{d/2}}\frac{1}{(2\pi s)^d}\int_D\int_A\int_A\exp(-|x-w|^2/2(t-s))$$
$$\times\exp(-|w-y_1|^2/2s)\exp(-|w-y_2|^2/2s)\,dy_1\,dy_2\,dw\,ds. \tag{4.18}$$

Let

$$f(t, \lambda) = \int_0^t C_{il}(t-s)\, C_{k_1 j_1}(s)\, C_{k_2 j_2}(s)$$

$$\times \int_D \int_A \int_A p(t-s, x, w)\, p(s, w, y_1)\, p(s, w, y_2)\, dy_1\, dy_2\, dw\, ds.$$

Thus, the behaviour of P_2 is studied through the behaviour of $f(t, \lambda)$.

The three cases are as follows:

(a) Subcritical process ($\lambda < 0$):

$$(4.16) \Rightarrow E(N(A, j), t \mid (x, i)) \to 0.$$

$$\lim_{t\to\infty} f(t, \lambda) = \lim_{t\to\infty} e^{\lambda t} \int_0^t e^{\lambda s}(e^{-\lambda(t-s)} C_{il}(t-s)\, e^{-\lambda s} C_{k_1 j_1}(s)\, e^{-\lambda s} C_{k_2 j_2}(s))$$

$$\times \int_D \int_A \int_A p(t-s, x, w)\, p(s, w, y_1)\, p(s, w, y_2)\, dy_1\, dy_2\, dw\, ds$$

$$\leqslant K \lim_{t\to\infty} e^{\lambda t} \int_0^t e^{\lambda s} \cdot 1\, ds, \qquad \text{where } K \text{ is a constant,}$$

$$= 0.$$

Therefore

$$(4.17) \Rightarrow \operatorname{Cov}(N((A, j_1), t \mid (x, i)), N((A, j_2), t \mid (x, i))) \to 0 \text{ as } t \to \infty.$$

(b) Critical process ($\lambda = 0$):

$$(4.16) \Rightarrow E(N(A, j), t \mid (x, i)) \to 0.$$

$$\lim_{t\to\infty} f(t, \lambda) \leqslant K \int_0^t \frac{1}{(2\pi(t-s))^{d/2}} \frac{1}{(2\pi s)^d} \int_D \int_A \int_A \exp(-|x-w|^2/2(t-s))$$

$$\times \exp(-|w-y_1|^2/2s) \exp(-|w-y_2|^2/2s)\, dy_1\, dy_2\, dw\, ds$$

$$= K \lim_{t\to\infty} \int_A \int_A \int_0^t \frac{1}{(\pi(2t-s))^{d/2}} \frac{1}{(4\pi s)^{d/2}}$$

$$\times \exp\left[- \left| x - \frac{(y_1+y_2)}{2} \right|^2 \Big/ (2t-s) \right]$$

$$\times \exp[-|y_1 - y_2|^2/4s]\, ds\, dy_1\, dy_2,$$

where K is a constant,

$$\leqslant K' \lim_{t\to\infty} \frac{1}{(\pi t)^{d/2}} \int_0^t \int_A \int_A \frac{1}{(4\pi s)^{d/2}} \times \exp[-|y_1 - y_2|^2/4s]\, ds\, dy_1\, dy_2,$$

where K' is a constant.

For $d = 1$,

$$\lim_{t\to\infty} f(t,\lambda) \leqslant \frac{K' |A|^2}{(2\pi)} \lim_{t\to\infty} \frac{1}{t^{1/2}} \int_0^t \frac{1}{s^{1/2}}\, ds = \frac{K' |A|^2}{\pi}.$$

For $d = 2$,

$$\lim_{t\to\infty} f(t,\lambda) \leqslant \frac{K' |A|}{\pi} \lim_{t\to\infty} \frac{1}{t} \int_0^t ds = \frac{K' |A|}{(2\pi)^2}.$$

For $d \geqslant 3$,

$$\lim_{t\to\infty} f(t,\lambda) \leqslant \frac{K' |A|}{2\pi^{d/2}} \lim_{t\to\infty} \frac{t}{t^{d/2}} = 0.$$

Therefore, $(4.17) \Rightarrow \mathrm{Cov}(N((A, j_1), t \mid (x, i)), N((A, j_2), t \mid (x, i)))$ remains bounded as $t \to \infty$ for $d = 1$ and 2, and converges to 0 for $d \geqslant 3$.

(c) Supercritical process ($\lambda > 0$):

$$(4.16) \Rightarrow E(N(A, j), t \mid (x, i)) \to \infty.$$

$$\lim_{t\to\infty} f(t,\lambda)$$

$$= \lim_{t\to\infty} e^{\lambda t} \int_0^t e^{\lambda s}(e^{-\lambda(t-s)} C_{il}(t-s)\, e^{-\lambda s} C_{k_1 j_1}(s)\, e^{-\lambda s} C_{k_2 j_2}(s)) \times \int_A \int_A \frac{1}{(\pi(2t-s))^{d/2}} \frac{1}{(4\pi s)^{d/2}} \exp\left[-\left|x - \frac{(y_1+y_2)}{2}\right|^2 \Big/ (2t-s)\right] \times \exp[-|y_1 - y_2|^2/4s]\, dy_1\, dy_2\, ds$$

$$> \lim_{t\to\infty} \frac{e^{\lambda t}}{(\sqrt{8\pi t})^d} \int_0^t e^{\lambda s}(e^{-\lambda(t-s)} C_{il}(t-s) e^{-\lambda s} C_{k_1 j_1}(s)\, e^{-\lambda s} C_{k_2 j_2}(s)) \times \int_A \int_A \exp\left[-\left\{\left|x - \frac{(y_1+y_2)}{2}\right|^2 + \left|\frac{y_1 - y_2}{4}\right|^2\right\} \Big/ s\right] dy_1\, dy_2\, ds.$$

Choose y_1^*, $y_2^* \in A$ such that

$$f^* = \left| x - \frac{(y_1^* + y_2^*)}{2} \right|^2 + \left| \frac{y_1^* - y_2^*}{4} \right|^2 \qquad \text{is a maximum.}$$

Then

$$\lim_{t\to\infty} f(t,\lambda) > \lim_{t\to\infty} \frac{|A|^2 e^{\lambda t}}{(\sqrt{8\pi t})^d} \int_0^t e^{\lambda s} e^{f^*/s} e^{-\lambda(t-s)} C_{il}(t-s)$$
$$\times e^{-\lambda s} C_{k_1 j_1}(s) e^{-\lambda s} C_{k_2 j_2}(s)\, ds \to \infty \qquad \text{since } \lambda > 0.$$

Therefore, (4.17) $\Rightarrow \operatorname{Cov}(N((A, j_1), t \mid (x,i)), N((A, j_2), t \mid (x,i))) \to \infty$.

5. The Multitype Branching Diffusion with Immigration

The multitype branching diffusion with immigration (MBDI) is a stochastic point process in which particles immigrate into $D = \mathbb{R}^d$ from an outside source and then diffuse and branch in D according to the laws of the MBD described in Section 4. To be precise, at $t = 0$, D is empty. A particle of type i appears in an element of volume dx in $(t, t+dt)$ with probability $r_i\, dx\, dt$; r_i is the immigration rate of particles of type i, and the immigration of the different types of particles is independent. The migration need not be Brownian motion, if Condition 4(a) is satisfied.

THEOREM 5.1. *Let $N^I(\cdot, t)$ denote the point process associated with the MBDI and assume that $E^i(j) < \infty$, $i = 1,\ldots, n$, $j = 1,\ldots, n$. Then the PGF of $N^I(\cdot, t)$, denoted by $G^I(\cdot, t)$, satisfies*

$$G^I(\phi, t) = \exp\left[-\sum_1^n r_i \int_0^t \int_D (1 - G(\phi, s \mid (x,i))\, dx\, ds \right], \quad 1 - \phi \in C_c(D'). \tag{5.1}$$

Proof. It will be assumed that $r_j = 0$, $j \neq i$, initially. If it can be shown that

$$G^I(\phi, t) = \exp\left[-r_i \int_0^t \int_D (1 - G(\phi, s \mid (x,i))\, dx\, ds \right], \tag{5.2}$$

(5.1) then follows immediately, since if more than one type of particle is immigrating, the resulting point process is a superposition of independent point processes, and the corresponding PGF is the product of the individual PGF's.

The proof of Theorem 5.2.1 of Dawson and Ivanoff [4] shows that if

$r_j = 0, j \neq i$, then if it exists, G^I satisfies (5.2). It must be shown that (5.2) is a PGF. Equation (5.2) may be written as

$$G^I(\phi, t) = \exp\left[\int_{\mathcal{N}'}\left(\exp\left(\sum_{j=1}^{n}\int_D \log\phi(x,j)\,N(dx,j)\right) - 1\right)\tilde{P}_i(dN,t)\right],$$

where $\tilde{P}_i(\mathscr{A}, t) = \int_0^t \int_D r_i P(\mathscr{A}, s \mid (x,i))\,dx\,ds$, $\mathscr{A} \in \mathscr{B}(\mathcal{N}')$, $\mathscr{A} \neq \varnothing$, $\tilde{P}_i(\varnothing, t) = 0$. $P(\,\cdot\,, s \mid (x,i))$ is the transition probability function for the MBD.

By Proposition 3.1, it is sufficient to show that $\tilde{P}_i(N(A,j) > 0, t) < \infty$, for any bounded $A \in \mathscr{B}(D)$. The expected number of type j descendents of i alive at time s is $C_{ij}(s)$. $E^i(j) < \infty$, $1 \leqslant i, j \leqslant n$, implies that $C_{ij}(\,\cdot\,)$ is a continuous function of t, which is bounded in finite intervals. Let

$$B_{ij}(t) = \sup_{0 \leqslant s \leqslant t} C_{ij}(s).$$

$$P(N(A,j) > 0, s \mid (x,i)) \leqslant C_{ij}(s)\int_A p(s,x,y)\,dy$$

$$\begin{aligned}\Rightarrow \tilde{P}_i(N(A,j) > 0, t) &\leqslant r_i \int_0^t C_{ij}(s)\int_A\int_D p(s,x,y)\,dx\,dy\\ &\leqslant r_i B_{ij}(t)\,|A|\,t\\ &< \infty.\end{aligned}$$

This completes the proof. ∎

The factorial cumulant densities are found easily by differentiating $\log G^I(\,\cdot\,, t)$:

$$\begin{aligned}&Q_k^I(t, (y_1, j_1), \ldots, (y_k, j_k))\\ &\quad = \sum_{i=1}^{n} r_i \int_0^t \int_D P_k(s, (y_1, j_1), \ldots, (y_k, j_k) \mid (x,i))\,dx\,ds.\end{aligned} \tag{5.3}$$

Using the usual moment–cumulant relationships,

$$\begin{aligned}P_1^I(t, (y,j)) &= \sum_{i=1}^{n} r_i \int_0^t \int_D P_1(s, (y,j) \mid (x,i))\,dx\,ds\\ &= \sum_{i=1}^{n} r_i \int_0^t \int_D C_{ij}(s)\,p(s,x,y)\,dx\,ds\\ &= \sum_{i=1}^{n} r_i \int_0^t C_{ij}(s)\,ds;\end{aligned} \tag{5.4}$$

$$\begin{aligned}
&P_2^I(t, (y_1, j_1), (y_2, j_2)) \\
&\quad = P_1^I(t, (y_1, j_1))\, P_1^I(t, (y_2, j_2)) \\
&\qquad + \sum_{i=1}^{n} r_i \int_0^t \int_D P_2(s, (y_1, j_1), (y_2, j_2) \mid (x, i))\, dx\, ds \\
&\quad = \left(\sum_{i=1}^{n} r_i \int_0^t C_{ij_1}(s)\, ds\right) \cdot \left(\sum_{i=1}^{n} r_i \int_0^t C_{ij_2}(s)\, ds\right) \\
&\qquad + \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} r_i V_l E^l(k_1, k_2) \\
&\qquad \times \int_0^t \int_D \int_0^s \int_D C_{il}(s-u)\, p(s-u, x, w) \\
&\qquad \times C_{k_1 j_1}(u)\, C_{k_2 j_2}(u)\, p(u, w, y_2)\, dw\, du\, dx\, ds \\
&\quad = \left(\sum_{i=1}^{n} r_i \int_0^t C_{ij_1}(s)\, ds\right) \left(\sum_{i=1}^{n} r_i \int_0^t C_{ij_2}(s)\, ds\right) \\
&\qquad + \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} r_i V_l E^l(k_1, k_2) \\
&\qquad \times \int_0^t \int_0^t C_{il}(v)\, C_{k_1 j_1}(u)\, C_{k_2 j_2}(u)\, p(2u, y_1, y_2)\, du\, dv. \qquad (5.5)
\end{aligned}$$

We now consider the limiting values as $t \to \infty$ of $E(N^I(A, j), t)$ and $\mathrm{Cov}(N^I((A, j_1), t), N^I((A, j_2), t))$ for a bounded set $A \in \mathscr{B}(D)$;

(a) Subcritical case ($\lambda < 0$):

$$(5.4) \Rightarrow E(N^I(A, j), t) = \sum_{i=1}^{n} r_i |A| \int_0^t C_{ij}(s)\, ds \to \sum_{i=1}^{n} r_i |A|\, C_{ij} \quad \text{as} \quad t \to \infty,$$

where $C_{ij} = \int_0^\infty C_{ij}(s)\, ds$ is a finite constant, by the Frobenius theorem.

Similarly, since $\int_A \int_A p(2u, y_1, y_2)\, dy_1\, dy_2 < |A|\ \forall u$, (5.4) and (5.5) show that as $t \to \infty$,

$$\begin{aligned}
\mathrm{Cov}(N^I((A, j_1), t), N^I((A, j_2), t)) &\to \delta(j_1 - j_2) \sum_{i=1} r_i |A|\, C_{ij_1} \\
&+ \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} r_i V_l E^l(k_1, k_2)\, C_{il} \\
&\times \int_0^\infty C_{k_1 j_1}(u)\, C_{k_2 j_2}(u) \int_A \int_A p(2u, y_1, y_2)\, dy_1\, dy_2\, du,
\end{aligned}$$

where the infinite integral converges.

(b) Critical case ($\lambda = 0$) and (c) supercritical case ($\lambda > 0$): The Frobenius theorem shows easily that for both critical and supercritical branching

$$E(N^I((A, j), t)) \to \infty,$$

and

$$\operatorname{Cov}(N^I((A, j_1), t), N^I((A, j_2), t)) \to \infty \qquad \text{as} \quad t \to \infty.$$

The effect of the immigration is clear: the subcritical process does not die off as it does in the case in which there is one particle initially and no immigration. The explosion in the critical case is in total contrast to the situation in which there is one particle initially and no immigration.

The convergence of the first and second moments in the subcritical case suggests the existence of a limiting steady state random field, as there is in the univariate case (cf. Ivanoff [6]).

THEOREM 5.2. *As $t \to \infty$, the subcritical MBDI converges in distribution to a steady state.*

Note. No assumption has been made about the migration process.

Proof. It is necessary and sufficient to show that

$$\begin{aligned} \lim_{t \to \infty} G^I(\phi, t) &= G^I(\phi) \\ &= \exp\left[-\sum_1^n r_i \int_0^\infty \int_D (1 - G(\phi, s \mid (x, i)) \, dx \, ds\right] \end{aligned}$$

is a PGF, and that the corresponding point process is a steady state. The argument that a limiting point process is a steady state is identical to that in the proof of Theorem 4.2.1 in Ivanoff [6].

Referring to the proof of Theorem 5.1, since

$$G^I(\phi) = \exp\left[\sum_{i=1}^n \int_{\mathscr{N}} \left(\exp\left(\sum_{j=1}^n \int_D \log \phi(x, j) \, N(dx, j)\right) - 1\right) \tilde{P}_i(dN)\right],$$

where $\tilde{P}_i(\mathscr{A}) = \int_0^\infty \int_D r_i P(A, s \mid (x, i)) \, dx \, ds$, $\mathscr{A} \neq \varnothing$, $\tilde{P}_i(\varnothing) = 0$, it is sufficient to show that $\tilde{P}_i(N(A, j) > 0) < \infty$ for bounded $A \in \mathscr{B}(D)$, $i = 1, ..., n$.

$$\begin{aligned} \tilde{P}_i(N(A, j) > 0) &\leqslant r_i \int_0^\infty C_{ij}(s) \int_A \int_D p(s, x, y) \, dx \, dy \\ &= r_i C_{ij} |A|. \quad \blacksquare \end{aligned}$$

The existence of spatial central limit theorems may be proven by using (5.3) and the concept of B-mixing.

THEOREM 5.3. *Let $N^I(\cdot, t)$ be the point process associated with the MBDI at time t. If all moments of the branching process up to and including order k exist, then $N^I(\cdot, t)$ is B-mixing of order k, $k \leqslant 4$.*

Proof.

$$Q_1^I(t', (y, j)) = \sum_{i=1}^{n} r_i \int_0^{t'} C_{ij}(s)\, ds; \tag{5.6}$$

$$\begin{aligned}
&\int_D Q_2^I(t', (y_1, j_1), (y_2, j_2))\, dy_1 \\
&\quad = \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} r_i V_l E^l(k_1, k_2) \\
&\qquad \times \int_0^{t'} \int_0^{t} \int_D \int_D \int_D C_{il}(t-s)\, p(t-s, x, w)\, C_{k_1 j_1}(s) \\
&\qquad \times p(s, w, y_1)\, C_{k_2 j_2}(s)\, p(s, w, y_2)\, dx\, dy_1\, dw\, ds\, dt \\
&\quad = \sum_{i=1}^{n} \sum_{l=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} r_i V_l E^l(k_1, k_2) \\
&\qquad \times \int_0^{t'} \int_0^{t} C_{il}(t-s)\, C_{k_1 j_1}(s)\, C_{k_2 j_2}(s)\, ds\, dt;
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
&\int_D \int_D Q_3^I(t', (y_1, j_1), (y_2, j_2), (y_3, j_3))\, dy_1\, dy_2 \\
&\quad = \int_0^{t'} \sum_{i=1}^{n} \sum_{l=1}^{n} r_i V_l \int_0^{t} C_{il}(t-u) \\
&\qquad \times \Bigg\{ \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} E^l(k_1, k_2, k_3)\, C_{k_1 j_1}(u)\, C_{k_2 j_2}(u)\, C_{k_3 j_3}(u) \\
&\qquad + \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} E^l(k_1, k_2) \int_D p(t-u, x, w) \\
&\qquad \times \Bigg[C_{k_1 j_1}(u) \int_D \int_D P_2(u, (y_2, j_2), (y_3, j_3) \mid (w, k_2))\, dy_2 \\
&\qquad + C_{k_1 j_2}(u) \int_D \int_D P_2(u, (y_1, j_1), (y_3, j_3) \mid (w, k_2))\, dy_1 \\
&\qquad + C_{k_1 j_3}(u) \int_D p(u, w, y_3) \\
&\qquad \times \int_D \int_D P_2(u, (y_1, j_1), (y_2, j_2) \mid (w, k_2))\, dy_1\, dy_2 \Bigg] dw\, dx \Bigg\}\, du\, dt.
\end{aligned}$$

But

$$\int_D P_2(u, (y_1, j_1), (y_2, j_2) \mid (w, k_2)\, dy_2$$

$$= \sum_{l'} V_{l'} \sum_{m_1} \sum_{m_2} \int_0^u \int_D \int_D E^{l'}(m_1, m_2)\, C_{k_2 l'}(u-s)\, p(u-s, w, z)$$

$$\times C_{m_1 j_1}(s)\, C_{m_2 j_2}(s)\, p(s, z, y_1)\, p(s, z, y_2)\, dy_2\, dz\, ds$$

$$= \sum_{l'} \sum_{m_1} \sum_{m_2} V_{l'} E^{l'}(m_1, m_2)\, p(u, w, y_1)$$

$$\times \int_0^u C_{k_2 l'}(u-s)\, C_{m_1 j_1}(s)\, C_{m_2 j_2}(s)\, ds,$$

and

$$\int_D \int_D P_2(u, (y_1, j_1), (y_2, j_2) \mid (w, k_2))\, dy_1\, dy_2$$

$$= \sum_{l'} \sum_{m_1} \sum_{m_2} V_{l'} E^{l'}(m_1, m_2) \int_0^u C_{k_2 l'}(u-s) C_{m_1 j_1}(s) C_{m_2 j_2}(s)\, ds;$$

$$\therefore \int_D \int_D Q_3^l(t', (y_1, j_1), (y_2, j_2), (y_3, j_3))\, dy_1\, dy_2$$

$$= \sum_{i=1}^n \sum_{l=1}^n r_i V_l \int_0^{t'} \int_0^t C_i(t-u)$$

$$\times \Bigg[\sum_{k_1=1}^n \sum_{k_2=1}^n \sum_{k_3=1}^n E^l(k_1, k_2)\, C_{k_1 j_1}(u)\, C_{k_2 j_2}(u)\, C_{k_3 j_3}(u)$$

$$+ \sum_{k_1=1}^n \sum_{k_2=1}^n E^l(k_1, k_2)$$

$$\times \Bigg\{ C_{k_1 j_1}(u) \left[\sum_{l'} \sum_{m_1} \sum_{m_2} \int_0^u C_{k_2 l'}(u-s)\, C_{m_1 j_2}(s)\, C_{m_2 j_3}(s)\, ds \right]$$

$$+ C_{k_1 j_2}(u) \left[\sum_{l'} \sum_{m_1} \sum_{m_2} \int_0^u C_{k_2 l'}(u-s)\, C_{m_1 j_1}(s)\, C_{m_2 j_3}(s)\, ds \right]$$

$$+ C_{k_1 j_3}(u) \Bigg[\sum_{l'} \sum_{m_1} \sum_{m_2} \int_0^u C_{k_2 l'}(u-s)$$

$$\times C_{m_1 j_1}(s)\, C_{m_2 j_2}(s)\, ds \Bigg] \Bigg\} \Bigg]\, du\, dt. \tag{5.8}$$

Finally, using similar methods and (4.15),

$$\int_D \int_D \int_D Q_4^l(t', (y_1, j_1), (y_2, j_2), (y_3, j_3), (y_4, j_4))\, dy_1\, dy_2\, dy_3$$

$$= \sum_{i=1}^{n} \sum_{l=1}^{n} r_i V_l \int_0^{t'} \int_0^{t} C_{il}(t-u)$$

$$\times \Bigg\{ \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} E^l(k_1, k_2, k_3, k_4)\, C_{k_1j_1}(u)\, C_{k_2j_2}(u)\, C_{k_3j_3}(u)\, C_{k_4j_4}(u)$$

$$+ \sum_{k_1} \sum_{k_2} \sum_{k_3} E^l(k_1, k_2, k_3)$$

$$\times \Bigg[C_{k_1j_1}(u)\, C_{k_2j_2}(u) \sum_{l'} \sum_{m_1} \sum_{m_2} V_{l'} E^{l'}(m_1, m_2)$$

$$\times \int_0^u C_{k_3l'}(u-s)\, C_{m_1j_3}(s)\, C_{m_2j_4}(s)\, ds$$

$$+ \cdots + C_{k_1j_3}(u)\, C_{k_2j_4}(u) \sum_{l'} \sum_{m_1} \sum_{m_2} V_{l'} E^{l'}(m_1, m_2)$$

$$\times \int_0^u C_{k_3l'}(u-s)\, C_{m_1j_1}(s)\, C_{m_2j_2}(s)\, ds \Bigg]$$

$$+ \sum_{k_1} \sum_{k_2} E^l(k_1, k_2) \Bigg[\Bigg[\sum_{l_1} \sum_{m_1} \sum_{m_2} V_{l_1} E^{l_1}(m_1, m_2)$$

$$\times \int_0^u C_{k_1l_1}(u-s)\, C_{m_1j_1}(s)\, C_{m_2j_2}(s)\, ds \Bigg]$$

$$\times \Bigg[\sum_{l_2} \sum_{m_3} \sum_{m_4} V_{l_2} E^{l_2}(m_3, m_4) \int_0^u C_{k_2l_2}(u-s)\, C_{m_3j_3}(s)\, C_{m_4j_4}(s)\, ds \Bigg]$$

$$+ \cdots + \Bigg[\sum_{l_1} \sum_{m_1} \sum_{m_2} V_{l_1} E^{l_1}(m_1, m_2)$$

$$\times \int_0^u C_{k_1l_1}(u-s)\, C_{m_1j_1}(s)\, C_{m_2j_4}(s)\, ds \Bigg]$$

$$\times \Bigg[\sum_{l_2} \sum_{m_3} \sum_{m_4} V_{l_2} E^{l_2}(m_3, m_4)$$

$$\times \int_0^u C_{k_2l_2}(u-s)\, C_{m_3j_2}(s)\, C_{m_4j_3}(s)\, ds \Bigg] + \cdots + \Bigg] \Bigg\}\, du\, dt. \tag{5.9}$$

The remaining terms involve only lower-order moments of the branching process, and iterated integrals involving the C_{ij}'s.

Examination of (5.6)–(5.9) immediately gives the desired result. ∎

In fact, it is straightforward but tedious to show that the existence of all moments up to and including order k of the branching process is necessary and sufficient for the MBDI to be B-mixing of order k.

The following corollaries are immediate results of the fact that the MBDI is infinitely divisible, and of Theorems 3.1 and 3.2.

COROLLARY 5.1. *Let $M_K^I(\cdot, t)$ be the renormalized version of $N^I(\cdot, t)$ defined in Theorem* 3.1. *If all moments of the branching process up to and including order* 3 *exist, then as $K \to \infty$, $M_K^I(\cdot, t)$ converges in law to the generalized n-variate Gaussian random field on D with covariance kernel*

$$\Gamma((x, i), (y, j); t) = \delta(x - y)\, \xi_2^t(i, j) + \xi_1^t(i)\, \delta(i - j)\, \delta(x - y),$$

where

$$\xi_1^t(i) = Q_1^I(t, (y, i)) = \sum_j r_j \int_0^t C_{ji}(s)\, ds, \tag{5.10}$$

$$\begin{aligned} \xi_2^t(i, j) &= \int_D Q_2^I(t, (y_1, i), (y_2, j))\, dy_1 \\ &= \sum_{m=1}^n \sum_{l=1}^n \sum_{k_1=1}^n \sum_{k_2=1}^n r_m V_l E^l(k_1, k_2) \\ &\quad \times \int_0^t C_{ml}(u) \int_0^{t-u} C_{k_1 i}(s)\, C_{k_2 j}(s)\, ds\, du. \end{aligned} \tag{5.11}$$

COROLLARY 5.2. *If $X_K^I(\mathbf{t}'; s)$ is defined using $M_K^I(\cdot, s)$ in* (3.6) *and if all moments of the branching process up to and including order* 4 *exist, then as $K \to \infty$, $X_K^I(\cdot\,; s)$ converges weakly in the Skorokhod topology to $G^s(\cdot)$, the multivariate Gaussian process with independent increments; $G^s(\mathbf{t}') = 0$ and $G^s(\mathbf{t}')$ has covariance matrix $\Gamma(s)\, t_1' \times \cdots \times t_d'$, where*

$$\Gamma(s) = \begin{bmatrix} \xi_1^s(1) + \xi_2^s(1, 1) & \xi_2^s(1, 2) & \cdots & \xi_2^s(1, n) \\ \xi_2^s(1, 2) & \xi_1^s(2) + \xi_2^s(2, 2) & & \xi_2^s(2, n) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_2^s(1, n) & \xi_2^s(2, n) & \cdots & \xi_1^s(n) + \xi_2^s(n, n) \end{bmatrix}$$

and $\xi_1^s(i)$ and $\xi_2^s(i, j)$ are defined by (5.10) *and* (5.11).

Finally, by examining (5.6)–(5.9), if the steady state of the subcritical MBDI is denoted by $N^I(\cdot)$, then since $C_{ij}(s) \sim u_i v_j e^{\lambda s}$, $N^I(\cdot)$ is mixing of order k if and only if all moments of the branching process up to and including order k exist. Thus, Corollaries 5.1 and 5.2 are also true for $N^l(\cdot)$, with

$$\xi_1^\infty(i) = \sum_j r_j \int_0^\infty C_{ji}(s)\, ds$$

and

$$\xi_2^\infty(i,j) = \sum_m \sum_l \sum_{k_1} \sum_{k_2} r_m V_l E^l(k_1, k_2) \times \int_0^\infty C_{ml}(s)\, ds \int_0^\infty C_{k_1 i}(u)\, C_{k_2 j}(u)\, du.$$

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